Noncommutative differential calculus for a D-brane in a nonconstant B field background with $H=0$

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In this paper we try to construct noncommutative Yang-Mills theory for generic Poisson manifolds. It turns out that the noncommutative differential calculus defined in an old work is exactly what we need. Using this calculus, we generalize results about the Seiberg-Witten map, the Dirac-Born-Infeld action, the matrix model and the open string quantization for constant B field to a nonconstant background with $H=0$.

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I. INTRODUCTION

For a D-brane in a constant B field background, its low energy effective theory lives on a noncommutative space for which the commutator

$$[x^a, x^b]_* = i\theta^{ab} \tag{1}$$

is a constant [1]. In the zero slope limit of Seiberg and Witten [2],

$$\alpha' \sim \epsilon^{1/2}, \quad g_{ab} \sim \epsilon, \tag{2}$$

where $\epsilon \to 0$, and B_{ab} is fixed, we have $\theta = B^{-1}$. If the B field background is not constant, but has a vanishing field strength ($H=dB$) for longitudinal directions on the D-brane, θ satisfies the Jacobi identity and so it defines a Poisson structure on the D-brane [3–5]. One can then use Kontsevich's formula [6] to define the *-product as a way to quantize the Poisson structure, and to define the noncommutative D-brane world volume. If $H \neq 0$, Kontsevich's formula can still be used to construct integrations to reproduce open string correlation functions [4]; however, the resulting algebra is nonassociative. An associative algebra was found by quantizing the open string [3], and it can also be used to reproduce the open string correlation functions [7].

In this paper we try to construct the noncommutative Yang-Mills theory for nonconstant B with $H=0$. For simplicity, we ignore terms of order θ^2 or higher; that is, we work in the Poisson limit. Interestingly, it turns out that an old work [8] on the formulation of differential calculus on noncommutative spaces is precisely what is needed for our purpose, i.e., to define the Seiberg-Witten map, and to find a noncommutative Yang-Mills (NCYM) theory that approximates the classical Dirac-Born-Infeld (DBI) action with the B field background.

The first problem one meets when trying to define a field theory on noncommutative space is to define derivatives. When θ is constant, we usually assume that

$$[\partial_a, x^b]_* = \delta_a^b. \tag{3}$$

However, when θ is not constant, this relation is inconsistent with Eq. (1), as can be easily seen by checking the Jacobi identity for (∂_a, x^b, x^c).

Another way to state the same problem is to notice that although we can define differential forms (dx^a) satisfying

$$[x^a, dx^b]_* = 0 \tag{4}$$

when θ is constant, after a general coordinate transformation this relation no longer holds, even for those transformations which keep θ constant. Thus even for a constant θ (which you can always achieve by coordinate transformations), it is unclear how to determine $[x^a,(dx^b)]_*$ or $[\partial_a, x^b]_*$ for a generic curved space.

One might suspect that, in the *-product representation of the algebra, the classical derivatives ∂_a can be used as the natural representative of the derivatives on noncommutative space. Although this is an algebraically consistent choice, we find that this definition is not enough. If the gauge transformation for a gauge potential is defined as

$$\hat{\delta}\hat{A}_a = \partial_a \hat{\lambda} - i[\hat{A}_a, \hat{\lambda}]_*, \tag{5}$$

the field strength defined by

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i[\hat{A}_a, \hat{A}_b]_* \tag{6}$$

will not be covariant. The reason is basically that the Leibniz rule fails

$$\partial_a(f*g) \neq (\partial_a f)*g + f*(\partial_a g). \tag{7}$$

In the context of matrix model, the generalized notion of covariant derivative, the covariant coordinates $X=x+A$, was introduced [9,2]. In particular the problems mentioned above were resolved in [10,11] by using only the covariant coordinates without referring to the derivatives. Thus although some results of this paper overlap with [11,12], here we are interested in a more conventional description of ncommutative gauge field theory, because this subject is of interest by itself apart from its application in string theory.

We review our mathematical formulation of noncommutative differential calculus at the Poisson level in Sec. II. In Sec. III we define the NCYM action, find the Seiberg-Witten map, and check that after suitable modification, the NCYM action agrees with the DBI action. In Sec. IV we provide an

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interpretation of the noncommutative calculus from the viewpoint of open string quantization. We also comment on the issue of background independence and general coordinate transformations.

II. POISSON DIFFERENTIAL CALCULUS

In this section we briefly summarize part of the results of [8].

To discuss differential calculus on a noncommutative space at the lowest order approximation, we define the Poisson structure on differential forms as follows. (Its dual description in terms of derivatives will be given in Sec. IV A.) The differential calculus on a noncommutative space is generated by the coordinates x and one-forms dx. We generalize the Poisson brackets on x,

$$(x^a,x^b)=\theta^{ab}(x), \tag{8}$$

to any two differential forms such that

$$(\omega_1,\omega_2)=(-1)^{p_1p_2+1}(\omega_2,\omega_1), \tag{9}$$

where p_i is 0 or 1 if ω_i is an even or odd form. After quantization, the Poisson brackets are an approximation to the commutator or anticommutator, depending on whether the two differential forms commute or anti commute classically. The *-product to the lowest order in θ is

$$\omega_1^*\omega_2=\omega_1\omega_2+\frac{i}{2}(\omega_1,\omega_2)+\cdots. \tag{10}$$

The associativity of the quantized algebra implies

$$\sum_{(i,j,k)}(-1)^{p_ip_k}(\omega_i,(\omega_j,\omega_k))=0, \tag{11}$$

where we sum over cyclic permutations of (i,j,k). For obvious reasons we also have the following requirements:

$$(\omega_1,\omega_2\omega_3)=(\omega_1,\omega_2)\omega_3+(-1)^{p_1p_2}\omega_2(\omega_1,\omega_3). \tag{12}$$

$$d(\omega_1,\omega_2)=(d\omega_1,\omega_2)+(-1)^{p_1}(\omega_1,d\omega_2). \tag{13}$$

Finally, we require that if ω_i is a n_i-form, (ω_1,ω_2) is always a (n_1+n_2)-form.

Assuming that θ^{ab} is invertible, we can always write (x,dx) in the form

$$(x^a,dx^b)=-\theta^{ac}\Gamma^b_{cd}dx^d, \tag{14}$$

where Γ^b_{cd} are some functions of x which transforms like a connection under general coordinate transformations. As Γ is in general not symmetric, one can use

$$\tilde{\Gamma}^a_b\equiv\Gamma^a_{bc}dx^c \quad \text{and} \quad \Gamma^a_b\equiv dx^c\Gamma^a_{cb} \tag{15}$$

as the connection one-forms to define two kinds of covariant derivatives $\tilde{\nabla}$ and ∇, respectively. Given θ and Γ, all Poisson brackets are determined.

Equation (13) is satisfied if and only if the Poisson structure is covariantly constant

$$\tilde{\nabla}_a\theta^{bc}\equiv\partial_a\theta^{bc}+\theta^{bd}\Gamma^c_{da}+\Gamma^b_{da}\theta^{dc}=0. \tag{16}$$

Intriguingly, we will see in Sec. III B that the existence of the Seiberg-Witten map requires that Eq. (16) holds.

It was found [8] that locally we can always make a change of coordinates such that for the new coordinates $\Phi^a=\Phi^a(x)$

$$(\Phi^a,\Phi^b)\equiv P^{ab}=\frac{1}{2}\tilde{R}^{ab}_{cd}\Phi^c\Phi^d+T^{ab}_c\Phi^c+\theta_0^{ab}, \tag{17}$$

where $\tilde{R},T$ and θ are constant tensors constrained by the Jacobi identity for P. For instance, $\tilde{R}$ has to satisfy the classical Yang-Baxter relation. Therefore, we can classify all Poisson differential calculus by these constant tensors up to linear transformations of Φ. Letting $x^a=\Phi^a$, one also finds [8] that

$$\Gamma^a_{bc}=P^{ad}\partial_bP^{-1}_{dc}. \tag{18}$$

Since the connection Γ is a pure gauge, it is convenient to define a new basis of one-forms

$$e_a=P^{-1}_{ab}d\Phi^b, \tag{19}$$

for which the algebra is greatly simplified

$$(e_a,x^b)=0, \quad (e_a,e_b)=-\frac{1}{2}\tilde{R}^{cd}_{ab}e_ce_d. \tag{20}$$

The constant tensor $\tilde{R}$, which also appeared in Eq. (17), is just the curvature tensor expressed in the basis of e_a for the connection one-form $\tilde{\Gamma}$, with its index raised by P. Similarly, T can be understood as the torsion at the origin $\Phi=0$.

An interesting feature of the Poisson differential calculus is that one can always realize the action of the exterior derivative on any function $f(x)$ through a one-form ξ

$$\xi=-e_a\Phi^a \tag{21}$$

as

$$df=(\xi,f)=-e_a(\Phi^a,f). \tag{22}$$

This is a property that will be useful for the formulation of matrix model.

The integration $\langle f\rangle$ of a function f over the noncommutative space can be defined as

$$\langle f\rangle\equiv\mathrm{Tr}(f), \tag{23}$$

where the trace is taken over a Hilbert space representation of the algebra of x after quantization. For constant θ,

$$|\mathrm{Pf}(\theta)|\mathrm{Tr}\leftrightarrow\int d^Dx, \tag{24}$$

where $\mathrm{Pf}(\theta)=\sqrt{\det\theta}$. Under general coordinate transformations $x\rightarrow x'$, $|\mathrm{Pf}(\theta)|^{-1}$ transforms by a factor of the Jacobian $|\det(\partial x'/\partial x)|$, so we can deduce that

$$\mathrm{Tr}\leftrightarrow\int d^Dx|\mathrm{Pf}(\theta^{-1})| \tag{25}$$

even when θ is not constant. It is easy to see that the integration $\langle\cdot\rangle$ is cyclic in the Poisson limit

$$\langle f*g\rangle=\langle g*f\rangle\Rightarrow\langle(f,g)\rangle=0 \tag{26}$$

for arbitrary integrable functions f and g. This implies that

$$\langle f*g\rangle\simeq\langle fg\rangle. \tag{27}$$

III. NONCOMMUTATIVE GAUGE THEORY

A. Noncommutative gauge fields

In this paper, we will mostly work at the Poisson level, that is, to the first order of θ or P. [The Γ defined in Eq. (14) is of the zeroth order.] Define the gauge transformation for a noncommutative gauge potential one-form

$$\hat{A}=dx^a\hat{A}_a(x) \tag{28}$$

by

$$\hat{\delta}\hat{A}=d\hat{\lambda}-i[\hat{A},\hat{\lambda}]_*\simeq d\hat{\lambda}+(\hat{A},\hat{\lambda}). \tag{29}$$

It follows from this that

$$\hat{\delta}\hat{A}_a\simeq\partial_a\hat{\lambda}+\theta^{bc}\nabla_b\hat{A}_a\partial_c\lambda, \tag{30}$$

where

$$\nabla_a\hat{A}_b\equiv\partial_a\hat{A}_b-\Gamma^c_{ab}\hat{A}_c. \tag{31}$$

Define the field strength two-form by

$$\hat{F}=d\hat{A}-i\hat{A}*\hat{A}=\frac{1}{2}dx^adx^b\hat{F}_{ab}(x). \tag{32}$$

For $U(1)$ gauge fields,

$$\hat{F}\simeq d\hat{A}+\frac{1}{2}(\hat{A},\hat{A}). \tag{33}$$

More explicitly,

$$\hat{F}_{ab}=\partial_a\hat{A}_b-\partial_b\hat{A}_a+\frac{1}{2}\nabla_c\hat{A}_a\theta^{cd}\nabla_d\hat{A}_b-\frac{1}{2}\tilde{\mathcal{R}}^{cd}_{ab}\hat{A}_c\hat{A}_d, \tag{34}$$

where

$$\tilde{\mathcal{R}}^{cd}_{ab}=\theta^{ce}\tilde{\mathcal{R}}^{d}_{eab}, \tag{35}$$

and

$$\tilde{\mathcal{R}}^a_b=\frac{1}{2}\tilde{\mathcal{R}}^a_{bcd}dx^cdx^d=d\tilde{\Gamma}^a_b+\tilde{\Gamma}^a_c\tilde{\Gamma}^c_b. \tag{36}$$

Under a gauge transformation,

$$\hat{\delta}\hat{F}\simeq(\hat{F},\hat{\lambda}), \tag{37}$$

where we used the fact that the Leibniz rule (13) holds. In terms of the components

$$\hat{\delta}\hat{F}_{ab}\simeq\theta^{cd}\nabla_c\hat{F}_{ab}\partial_d\hat{\lambda}, \tag{38}$$

where

$$\nabla_a\hat{F}_{bc}\equiv\partial_a\hat{F}_{bc}-\hat{F}_{bd}\Gamma^d_{ac}-\Gamma^d_{ab}\hat{F}_{dc}. \tag{39}$$

We also need the Leibniz rule to show that $\hat{F}$ satisfies the Bianchi identity

$$d\hat{F}+(\hat{A},\hat{F})\simeq0. \tag{40}$$

The covariant differential one-form is defined as

$$\hat{D}=d+\hat{A}=dx^a(\partial_a+\hat{A}_a). \tag{41}$$

One may also choose to use e_a as the basis of one-forms. Using Eq. (21), we find

$$\hat{D}\simeq\xi+A=e_a(-\Phi^a+\hat{A}^a) \tag{42}$$

when acting on functions of x. So the covariant coordinate $X=x+A$ has a conventional interpretation when x happens to be the canonical coordinate Φ.

B. Seiberg-Witten map

The Seiberg-Witten map is a map from a commutative gauge potential A to a noncommutative gauge potential $\hat{A}$ such that the gauge transformations of A are mapped to the gauge transformations of $\hat{A}$ [2]. The Seiberg-Witten map for arbitrary Poisson structure θ was given in [10] for the A field defined in the covariant coordinates $X=x+A$. Here we present a map for the A field defined in the covariant derivative $D=\partial+A$. This map has to match gauge transformations of A to those of $\hat{A}$

$$\hat{A}(A)+\hat{\delta}_{\hat{\lambda}}\hat{A}(A)=\hat{A}(A+\delta_\lambda A). \tag{43}$$

Here we only consider $U(1)$ gauge fields, so $\delta_\lambda A=d\lambda$.

Comparing with the Seiberg-Witten map for constant θ, we have a new term on the left-hand side of Eq. (43) by differentiating θ in $\hat{\lambda}$. We hope to compensate this change by choosing the Γ^a_{bc} factor in Eq. (14) as well as modifying the Seiberg-Witten map appropriately. Interestringly, it turns out that we should demand Γ to satisfy the Leibniz rule (16). (As we mentioned in the end of Sec. II, this guarantees that all Leibniz rules can be satisfied.) We modify the Seiberg-Witten map as

$$\hat{A}_a=A_a-\frac{1}{2}\theta^{bc}A_b(\nabla_cA_a+F_{ca})+\cdots. \tag{44}$$

The transformation parameters $\hat{\lambda}$ and λ are still related by the same map

$$\hat{\lambda}=\lambda+\frac{1}{2}\theta^{ab}\partial_a\lambda A_b+\cdots. \quad (45)$$

It follows from Eqs. (32) and (44) that

$$\hat{F}_{ab}\simeq F_{ab}-(F\theta F)_{ab}-A_c\theta^{cd}\nabla_d F_{ab}. \quad (46)$$

For an arbitrary scalar field $f(x)$ which is invariant under the gauge transformations, we can make it covariant by modifying it to be

$$\hat{f}\simeq f-A_a\theta^{ab}\partial_b f \quad (47)$$

so that

$$\hat{\delta}\hat{f}\simeq(\hat{f},\hat{\lambda}). \quad (48)$$

In general, for an arbitrary differential form ω invariant under gauge transformations,

$$\hat{\omega}_{abc\cdots}\simeq\omega_{abc\cdots}-A_d\theta^{de}\nabla_e\omega_{abc\cdots} \quad (49)$$

is covariant, i.e.,

$$\hat{\delta}\hat{\omega}\simeq(\hat{\omega},\hat{\lambda}). \quad (50)$$

The Seiberg-Witten map is a crucial part in showing the background independence of string theory. In the next section we will define the NCYM action and in Sec. III D we will use the Seiberg-Witten map to show that the DBI action with B field background is approximately equal to the NCYM action on the noncommutative space with $\theta\simeq B^{-1}$.

C. NCYM

In this section we construct the noncommutative Yang-Mills (NCYM) action.

Denote the metric on the noncommutative space by G. If G is not constant, we have to modify it to be a covariant metric for the NCYM action. Let

$$\nabla_a G^{bc}\equiv\partial_a G^{bc}+G^{bd}\Gamma^c_{ad}+\Gamma^b_{ad}G^{dc}, \quad (51)$$

then

$$\hat{G}^{ab}\equiv G^{ab}-A_c\theta^{cd}\nabla_d G^{ab}+\cdots \quad (52)$$

is covariant in the sense that it transforms like $\hat{F}$

$$\hat{\delta}\hat{G}^{ab}\simeq\theta^{cd}\nabla_c\hat{G}^{ab}\partial_d\hat{\lambda}. \quad (53)$$

Define the NCYM action by

$$S_{NCYM}=-\frac{1}{4g_{YM}^2}\langle\mathrm{tr}(\hat{G}*\hat{F}*\hat{G}*\hat{F})\rangle$$
$$=-\frac{1}{4g_{YM}^2}\langle\hat{G}^{ab}*\hat{F}_{bc}*\hat{G}^{cd}*\hat{F}_{da}\rangle. \quad (54)$$

To the lowest order in θ,

$$\hat{\delta}S_{NCYM}=-\frac{1}{4g_{YM}^2}\langle(\mathrm{tr}(\hat{G}\hat{F}\hat{G}\hat{F}),\hat{\lambda})\rangle=0. \quad (55)$$

So the NCYM action is invarinat under gauge transformations.

The NCYM action can be further simplified at the Poisson level. To do so we derive an identity that will also be used later. First consider the integration

$$\langle\mathrm{tr}((M,N)MN)\rangle, \quad (56)$$

where M,N are symmetric or antisymmetric matrices of functions. Using Eq. (26), we find

$$\langle\mathrm{tr}((M,N)MN)\rangle=\langle\mathrm{tr}((M,NMN)-\theta^{ab}\partial_a MN\partial_b MN$$
$$-\theta^{ab}\partial_a MNM\partial_b N)\rangle$$
$$=-\langle\mathrm{tr}(\theta^{ab}(\partial_b NMN\partial_a M)^T)\rangle$$
$$=-\langle\mathrm{tr}((M,N)MN)\rangle=0. \quad (57)$$

It follows from this that

$$\langle\mathrm{tr}(M*N*M*N)\rangle\simeq\left\langle\mathrm{tr}\left(\left(MN+\frac{i}{2}(M,N)\right)\right.\right.$$
$$\left.\left.*\left(MN+\frac{i}{2}(M,N)\right)\right)\right\rangle$$
$$=\left\langle\mathrm{tr}\left(MNMN+\frac{i}{2}(MN,MN)\right.\right.$$
$$\left.\left.+i(M,N)MN\right)\right\rangle$$
$$=\langle\mathrm{tr}(MNMN)\rangle. \quad (58)$$

[In fact, one can show that

$$\langle\mathrm{tr}(M*N*M*N')\rangle\simeq\langle\mathrm{tr}(MNMN')\rangle \quad (59)$$

for symmetric or antisymmetric matrices M,N,N'.] Hence, the NCYM action at the Poisson level is just

$$S_{NCYM}\simeq-\frac{1}{4g_{YM}^2}\langle\mathrm{tr}(\hat{G}\hat{F}\hat{G}\hat{F})\rangle. \quad (60)$$

D. NCYM and DBI

In the Seiberg-Witten limit, it is generally believed that the low energy effective field theory for a D-brane in the background of a nonconstant B field lives on a noncommutative space with the Poisson structure $1/B$ [13,3]. It is therefore natural to propose that the NCYM action describes the physics for the $U(1)$ gauge field on the D-brane. On the other hand, the DBI action is known to be the low energy effective action for slowly varying fields on a D-brane [14,15]. We should thus be able to match the NCYM action with the DBI action in the leading order in α' at least at the Poisson level. This was shown for constant B in [2,16], and for generic B in [11] in the language of covariant coordinates. Here we want to rederive this result for nonconstant B in the language of covariant derivatives.

Derivative corrections to the DBI action [17–19] is of order $\mathcal{O}(F^2(\partial F)^2)$ for bosonic string theory and of order $\mathcal{O}(F^2(\partial^2 F)^2)$ for superstring theory. These are subleading terms in α' and thus should be ignored. Since F can only appear via the gauge invariant combination $(B+F)$ in the D-brane action, this means that terms linear in (∂B) should cancel in the NCYM action.

The DBI action of a D-brane is

$$S_{DBI}=T_p\int d^Dx\sqrt{g+2\pi\alpha'(B+F)}, \tag{61}$$

where g is the closed string spacetime metric, and B is the Neveu-Schwarz–Neveu-Schwarz (NS-NS) B field background. The Dp-brane tension is

$$T_p=\frac{1}{(2\pi)^p g_s(\alpha')^{(p+1)/2}}. \tag{62}$$

We will identify the F in the DBI action with the commutative F related to $\hat{F}$ in the Seiberg-Witten map (44).

Using the identity

$$\begin{aligned}\sqrt{\det(1+M)}=&1+\frac{1}{2}\mathrm{tr}(M)+\frac{1}{8}(\mathrm{tr}(M))^2-\frac{1}{4}\mathrm{tr}(M^2)\\&+\frac{1}{48}(\mathrm{tr}(M))^3-\frac{1}{8}\mathrm{tr}(M)\mathrm{tr}(M^2)\\&+\frac{1}{6}\mathrm{tr}(M^3)+\mathcal{O}(M^4),\end{aligned} \tag{63}$$

one finds that the leading order terms in α' in the DBI Lagrangian are [2]

$$\begin{aligned}\mathcal{L}_{DBI}=&(2\pi\alpha')^{(p+1)/2}T_p|\mathrm{Pf}(B+F)|\\&-\frac{1}{4}(2\pi\alpha')^{(p+5)/2}T_p|\mathrm{Pf}(B)|(f_1+f_2),\end{aligned} \tag{64}$$

where G and θ are the symmetric and antisymmetric part of $(g+B)^{-1}$, i.e.,

$$\frac{1}{g+2\pi\alpha' B}=G+\frac{1}{2\pi\alpha'}\theta, \tag{65}$$

and

$$f_1=\left(1+\frac{1}{2}\mathrm{tr}(\theta F)\right)\mathrm{tr}(GFGF)-2\,\mathrm{tr}(\theta FGFGF)+\cdots, \tag{66}$$

$$f_2=\left(1+\frac{1}{2}\mathrm{tr}(\theta F)\right)\mathrm{tr}(GBGB)-2\,\mathrm{tr}(GBGF)+\cdots. \tag{67}$$

In f_1 and f_2 the omitted terms are of higher order in θF or ϵ in the Seiberg-Witten limit (2). Since $B^{-1}\sim\theta$, some terms omitted in f_2 are in fact of the same order as some of the terms in f_1 which we keep. However, as we have only defined the calculus at the Poisson level, we can only compare terms of the leading order in θ, and the background field B is viewed as of order $\mathcal{O}(\theta^0)$ in the sense of Poisson brackets. Our result in this section will be justified in Sec. III E by the background independence of the matrix model.

The first term in Eq. (64) is a total derivative for $dB=0$ and $B>F$ in the sense that $|Pf(B+F)|=Pf(B+F)$. The terms in f_2 are independent of F up to a total derivative if B and g are constant, but we need to keep it for nonconstant B.

Compared with the constant θ case, the NCYM for nonconstant θ is roughly speaking only modified by replacing all derivatives by the covariant derivatives ∇ as shown in Eqs. (46), (52). Using the Leibniz rule for the covariant derivative, e.g.,

$$\partial_a(f_b g^b)=(\nabla_a f_b)g^b+f_b(\nabla_a g^b), \tag{68}$$

we find after straightforward calculations that f_1 agrees with the NCYM action for the coupling constant

$$g_{YM}^2=(2\pi)^{(p-5)/2}g_s(\alpha')^{-2}. \tag{69}$$

In matching the two actions, there is actually an ambiguity in choosing whether the metric G in Eq. (55) is defined by $-B^{-1}*g*B^{-1}$ or just $-B^{-1}gB^{-1}$. In our approximation, however, there is no difference between these two choices.

To take care of f_2, we need to modify the NCYM action as

$$S_D=-\frac{1}{g_{YM}^2}\langle(\mathrm{tr}[\hat{G}*(\hat{F}+\hat{B})*\hat{G}*(\hat{F}+\hat{B})])\rangle, \tag{70}$$

where

$$\hat{B}_{ab}\equiv B_{ab}-A_c\theta^{cd}\nabla_d B_{ab}+\cdots. \tag{71}$$

Since

$$\hat{\delta}\hat{B}=(\hat{B},\hat{\lambda}), \tag{72}$$

S_D is gauge invariant. We will see later that this modification is needed for the NCYM action to have a matrix model interpretation.

Let us call the metric appearing in the fundamental string action the closed string metric, and the metric appearing in the NCYM action the open string metric [2]. It is amusing to see that for the closed string, g_{ab} is the metric for the basis dx^a, and $G^{ab}=-(B^{-1}gB^{-1})^{ab}$ is the metric for the basis e_a

$$ds^2_{closed}=g_{ab}dx^a dx^b=G^{ab}e_a e_b\,. \tag{73}$$

On the other hand, it is the opposite for the open string. $G_{ab}=-(Bg^{-1}B)_{ab}$ is the metric for the basis dx^a, and g^{ab} is the metric for the basis e_a

$$ds^2_{open}=g^{ab}e_a e_b=G_{ab}dx^a dx^b. \tag{74}$$

E. Matrix model

Due to Eq. (21), we expect that a matrix model expression for the NCYM is available for the basis of one-forms e_a. Let

$$\hat{A}=e_a\hat{A}^a, \quad \hat{F}=\frac{1}{2}e_a e_b\hat{F}^{ab}. \tag{75}$$

It can be checked that

$$\hat{F}^{ab}=(X^a,X^b)-P^{ab}(X), \tag{76}$$

where

$$X^a\equiv\Phi^a-\hat{A}^a. \tag{77}$$

This X is thus the so-called covariant coordinate [10].

In terms of X the action of the NCYM (70) can be written as

$$S_{NCYM}=\frac{1}{4g^2_{YM}}\langle\hat{g}_{ab}*[X^b,X^c]_* * \hat{g}_{cd}*[X^d,X^a]_*\rangle. \tag{78}$$

This is precisely the leading order term one expects for a matrix model in a curved space time [20]. The background independence of B for the NCYM action is manifest in this form, as in the case of constant B [2]. In particular, for the case of flat spacetime, we know that this expression is exact. One can start from this action to obtain the higher order terms in the NCYM action (70), using the Kontsevich formula to define the $*$-product.

IV. REMARKS

A. Dual description

Instead of using differential forms to describe differential calculus, we can just use functions and derivatives. These two descriptions are dual to each other. Starting with the Poisson differential calculus in the previous section, here we construct the noncommutative algebra for derivatives and functions in the way that has been used for many noncommutative spaces in the past [21].

Classically, when the exterior derivative d acts on functions, it is equivalent to $dx^a\partial_a$. Let us assume that this holds also for noncommutative spaces. For example, we have

$$dx^a=[dx^b\partial_b,x^a]_*\,. \tag{79}$$

At the Poisson level, it is

$$dx^a=-(dx^b p_b,x^a). \tag{80}$$

We will identify p with the conjugate momentum of x. It follows from this and Eq. (14) that

$$(x^a,p_b)=\delta^a_b+\theta^{ac}\Gamma^d_{cb}p_d\,. \tag{81}$$

Assume that the Leibniz rule (13) is observed, for an arbitrary function $f(x)$, the nilpotency of d $(dd=0)$ will be satisfied at the Poisson level if

$$(p_a,p_b)=-\frac{1}{2}r^{cd}_{ab}p_c p_d\,, \tag{82}$$

where r is defined by

$$(dx^a,dx^b)=-\frac{1}{2}r^{ab}_{cd}dx^c dx^d=-\frac{1}{2}d(\theta^{ac}\tilde{\Gamma}^b_c+\theta^{bc}\tilde{\Gamma}^a_c). \tag{83}$$

In the next subsection we will show that both relations (81) and (82) are results of quantization for an open string in the low energy limit.

B. Open string quantization

The bosonic action for an open string ending on a Dp-brane is given by

$$S=\int d^2\sigma(g_{\mu\nu}\partial_\alpha X^\mu\partial^\alpha X^\nu+2\pi\alpha'\epsilon^{\alpha\beta}B_{ab}\partial_\alpha X^a\partial_\beta X^b), \tag{84}$$

where $\mu,\nu=0,1,\ldots,9$, and $a,b=0,1,\ldots,p$, and g is the closed string metric. In the low energy limit, the string is very short and so one can approximate the string by a straight line

$$X(\sigma)=x+\sigma X'\,, \tag{85}$$

where $X'\simeq(X(\pi)-X(0))/\pi$ is very small. For $H=0$, the symplectic two-form is [3]

$$\Omega=\hat{\mathcal{F}}_{ab}(X)dX^a dX^b\big|^{\sigma=\pi}_{\sigma=0}\,, \tag{86}$$

where $\hat{\mathcal{F}}=B$ in the Seiberg-Witten limit. From this we can find the Poisson brackets among x and X', and we will keep terms of order $\mathcal{O}((X')^2)$ or lower, since we assumed that X' is small. This is equivalent to say that B is slowly varying compared with the length of the string, and so we will only keep terms up to $\mathcal{O}(\partial^2 B)$. One then obtains

$$\begin{aligned}(x^a,x^b)&=\theta^{ab}(x)\\(y^a,y^b)&=\theta^{ab}(y)\\&\simeq\theta^{ab}(x)+(y-x)^c\partial_c\theta^{ab}bf\\&\quad+\frac{1}{2}(y-x)^c(y-x)^d\partial_c\partial_d\theta^{ab},\end{aligned} \tag{87}$$

where $x\equiv X(\sigma=0)$ and $y\equiv X(\sigma=\pi)$.

The momentum associated with the endpoint at $\sigma=0$ is approximately

$$p_a \simeq \pi \theta^{-1}_{ab}(x) X'^b. \tag{88}$$

Remarkably, within our approximation we can calculate precisely the desired commutation relations (81) and (82) among x and p for the connection one-form

$$\Gamma^c_{ab} = \theta^{cd} \partial_a \theta^{-1}_{db}. \tag{89}$$

Comparing this with Eq. (18), we find that x here is already the canonical coordinate denoted Φ in Sec. II. Since we ignored terms of order $\partial^3 B$, θ is well defined only up to quadratic terms in x as our P in Eq. (17).

It follows from Eq. (89) that the Leibniz rule (13) is satisfied. It is interesting to note that *a priori* the Leibniz rule does not have to be satisfied from the viewpoint of open string quantization. (In the defining conditions for the Poisson differential calculus, only the Jacobi identities for functions are guaranteed by the associativity of a consistent quantization. In fact, the Jacobi identities among differential forms are not automatically satisfied either, unless the exterior derivative is an element of the algebra obtained from open string quantization.) Nevertheless the result of open string quantization is just sufficient to define the noncommutative gauge theory and the Seiberg-Witten map. The differential calculus defined in [8] happen to be a natural framework for describing a D-brane world volume in B field background.

The matrix model action (78) is greatly simplified when the metric g is constant (and so the space is flat) because $\hat{g} = g[\hat{g} = g(X) = g(\Phi - A)]$ for this case. For the connection (89), this implies that $\nabla G = 0$ and thus $\hat{G} = G$. Hence it is also the case when the NCYM action (54) is greatly simplified.

C. Conclusion

The property of background independence in B is shown to hold perfectly for the NCYM action in [2] for constant B. This is because the NCYM action for constant θ is equivalent to the matrix model action (78) [2]. Obviously that action is not changed under

$$[x^a, x^a] = i\theta^{ab} \rightarrow [x'^a, x'^b] = i\theta' \tag{90}$$

as long as $x + \hat{A} = x' + \hat{A}'$. (See also [22] for relevant discussions.) Since the NCYM action (70) can be rewritten as the matrix model action (78), obviously it is independent of the choice of Γ, as long as Eq. (16) is satisfied. [On the other hand, for the NCYM action (54) which does not have the background term, it can depend on the choice of Γ.] Note that this independence of Γ is actually associated with the symmetry of general coordinate transformations on the brane. If one quantizes the open string in two different coordinate systems and get two θ's and two Γ's, although the θ's will be related by a coordinate transformation, the Γ's will not. The fact that the D-brane theory should be invariant under coordinate transformations on the brane implies that the NCYM theory for D-brane should be independent of the choice of Γ. This does not mean that we do not need Γ—at least we need it to satisfy Eq. (16). Moreover, to deal with specific problems, some choices of Γ may be more convenient than others.

The NCYM action for D-branes with nonconstant background (70) may be useful even for the flat case. Note that for the quantization of the open string, only the gauge invariant quantity $B+F$ matters, although we have conventionally called the background part of $B+F$ the B field background. Nevertheless if F has a background value due to some source living on the D-brane, it will also make the D-brane world volume noncommutative. For configurations with a background field which varies greatly over the D-brane, such as the configurations corresponding to a semi-infinite string ending on the D-brane [23], we expect that our description may be more appropriate than the NCYM with constant θ, since a small fluctuation of that configuration is still a big fluctuation for any choice of constant background. [On the other hand, as we just mentioned, the action (78) is background independent, so nonperturbatively they must be equivalent.]

In this paper we constructed the noncommutative calculus suitable for the description of D-branes, and checked that the NCYM action matches with the DBI action at the Poisson level for $H=0$. It is natural to ask how these results can be generalized when $H \neq 0$. The (associative) noncommutative algebra for this case was found in [3]. The novelty of this algebra is that the commutation relations between functions of x is a function of both x and p. This poses a problem to the definition of gauge transformations on such noncommutative spaces, but this problem was solved in [7]. Hopefully a differential calculus can be constructed so that certain geometric understanding about these gauge symmetries can be obtained.

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